

Robustness of the in-degree exponent for the World-Wide Web

B. Kahng,¹ Y. Park,² and H. Jeong³

¹*School of Physics and Center for Theoretical Physics, Seoul National University, Seoul 151-747, Korea*

²*Department of Physics, Myongji University, Yongin, Kyunggi-do 449-728, Korea*

³*Department of Physics, Korea Advanced Institute of Science and Technology, Taejon 305-701, Korea*

(Received 14 December 2001; revised manuscript received 2 July 2002; published 10 October 2002)

We consider a stochastic model for directed scale-free networks following power laws in the degree distributions in both incoming and outgoing directions. In our model, the number of vertices grow geometrically with time with a growth rate p . At each time step, (i) each newly introduced vertex is connected to a constant number of already existing vertices with the probability linearly proportional to in-degree distribution of a selected vertex, and (ii) each existing vertex updates its outgoing edges through a stochastic multiplicative process with mean growth rate of outgoing edges g and its variance σ^2 . Using both analytic treatment and numerical simulations, we show that while the out-degree exponent γ_{out} depends on the parameters, the in-degree exponent γ_{in} has two distinct values, $\gamma_{\text{in}}=2$ for $p>g$ and 1 for $p<g$, independent of different parameters values. The latter case has logarithmic correction to the power law. Since the vertex growth rate p is larger than the degree growth rate g for the World-Wide Web (WWW) nowadays, the in-degree exponent appears robust as $\gamma_{\text{in}}=2$ for the WWW.

DOI: 10.1103/PhysRevE.66.046107

PACS number(s): 89.75.Hc, 89.70.+c, 87.18.Sn, 89.75.Da

I. INTRODUCTION

Complex system consists of many constituents such as individuals, substrates, and companies in social, biological, and economic systems, respectively, showing cooperative phenomena between constituents through diverse interactions and adaptations to the pattern they create [1,2]. Recently, there have been considerable efforts to understand such complex systems in terms of random graph, consisting of vertices and edges, where vertices (edges) represent constituents (their interactions). This approach was initiated by Erdős and Rényi (ER) [3]. In the ER model, the number of vertices is fixed, while edges connecting one vertex to another occur randomly with certain probability. The ER model is, however, too random to describe complex systems in real world.

An interesting feature emerging in such complex systems is the scale-free (SF) behavior in the degree distribution $P(k)\sim k^{-\gamma}$, where the degree k is the number of edges incident upon a given vertex. Barabási and Albert (BA) [4,5] introduced an evolving model illustrating SF network. In the BA model, the number of vertices increases linearly with time, and a newly introduced vertex is connected to m already existing vertices, following the so-called preferential attachment (PA) rule that the vertices with more edges are preferentially selected for the connection to the new vertex with the probability linearly proportional to the degree of that vertex. Then it is known that the degree distribution follows $P(k)\sim k^{-3}$ for the BA model. While the BA model is meaningful as the first step to generate SF network, it is too simple to be in accordance with real-world networks. Extended versions of the BA model have been introduced [6,7], taking into account additional local events such as adding new edges, or rewiring edges from one vertex to another. Depending on the frequency of these processes, the degree distribution either remains as SF with the exponent depend-

ing on the details of the local event or follows an exponential decay.

Huberman and Adamic (HA) [8] proposed another scenario for SF networks. In the HA model, the number of vertices grows geometrically with time, and edges of each vertex evolve following a stochastic multiplicative process. Combining these two ingredients leads to a power-law behavior in the degree distribution, where the exponent is determined by the growth rates of vertices, and the mean degree and the variance of the fluctuations arising in the stochastic process of updating edges. While the HA and BA models look fundamentally different at a first glance, they are similar in essence. One can show easily that the multiplicative process is reduced to the PA rule when the time dependence of the total number of edges is the same as that of the number of newly introduced vertices. Moreover, the stochastic process in the HA model might be related to the rewiring process in the extended model of the BA model [6].

SF networks may be classified into undirected or directed network whether directionality is assigned to edges or not. Typical examples of undirected networks include the actor network [9], the author collaboration network [10], and the internet with equal uploading and downloading rates [11]. Directed networks are also ubiquitous in real world such as the World-Wide Web (WWW) [8,12,13], the citation network of scientific papers [14], biological networks such as metabolic networks [15] and neural networks, *etc.* Recently, Albert *et al.* [12] and Huberman *et al.* [8] investigated the topology of the WWW extensively, and found that the in-degree and the out-degree distributions of the WWW exhibit power-law behaviors with different exponents, i.e., $P_{\text{in}}(k_{\text{in}})\sim k_{\text{in}}^{-\gamma_{\text{in}}}$ and $P_{\text{out}}(k_{\text{out}})\sim k_{\text{out}}^{-\gamma_{\text{out}}}$, respectively. Here the in-degree k_{in} (out-degree k_{out}) means the number of edges incident upon (emanating from) a given vertex. Further studies [13,16,17] showed that γ_{in} is robust as $\gamma_{\text{in}}\approx 2.1$ for different systems, while γ_{out} varies depending on systems, ranging

TABLE I. Numerical values of in-degree and out-degree exponents for various systems.

Refs.	System	In-degree exponent	Out-degree exponent
Ref. [12]	nd. edu	2.1	2.45
Ref. [13]		2.1	2.38
Ref. [17]	off-site	2.1	2.67
Ref. [17]	total	2.1	2.72

from 2.4 to ~ 2.7 as tabulated in the Table I.

Theoretical studies for directed networks have less been carried out compared with those for undirected networks. When the directionality is assigned to edge in the BA model, pointing from a new vertex to old ones, the in-degree and the out-degree distributions follow $P_{\text{in}}(k_{\text{in}}) \sim k_{\text{in}}^{-3}$ and $P_{\text{out}}(k_{\text{out}}) = \delta(k_{\text{out}} - m)$, respectively, which is not relevant to the empirical results for the WWW. Dorogovtsev and Mendes [18] performed a similar study using the rate equation, in which the in-degree distribution follows a power law whereas the out-degree distribution is of the δ function. More recently, Krapivsky *et al.* [19] studied directed SF networks using the rate equation method for the simple case similar to the one introduced by Tadić [20] that at each time step, a vertex is newly introduced and connected to an old vertex following the PA rule with a certain probability and an internal directed edge is connected between two vertices chosen following the PA rule with the remaining probability. They obtained the in-degree and the out-degree distributions analytically, both of which exhibit power-law behaviors with different exponents depending on the detail of the parameters they used. While their analytic treatment was successful in generating the empirical values of the out-degree and the in-degree exponents for the WWW by tuning the parameters, their model is unable to illustrate the robustness of the in-degree exponent for various systems because different parameters tune different values of γ_{in} and γ_{out} at the same time.

In this paper, we introduce a stochastic model for directed SF networks exhibiting power-law behaviors with distinct exponents in both incoming and outgoing directions and present an analytic solution for the model. Through this study, we can illustrate why the in-degree exponent is robust for different systems, while the out-degree exponent depends on the details of systems. This behavior occurs when the growth rate of the number of vertices is large enough compared with the effective growth rate of degree of each vertex.

This paper is organized as follows. In Sec. II, we will introduce a stochastic model. In Secs. III and IV, analytic solutions for the out-degree and the in-degree distributions will be presented, respectively. In Sec. V, we will present the result of numerical simulations for the model in the vertex growth dominant and the degree growth dominant regimes, respectively. The final section will be devoted to the conclusions.

II. THE MODEL

Let us introduce a directed SF network model as follows: (i) At each time step, the total number of vertices increases

geometrically with growth rate p , i.e.,

$$N(t) = N(t-1)(1+p). \quad (1)$$

So the total number of vertices newly introduced at time t is $pN(t-1)$. (ii) m edges emanate from each new vertex, pointing to m distinct old vertices following the PA rule. The probability to connect to a vertex j is given by

$$\Pi_{i \rightarrow j} = \frac{k_{\text{in},j}(t-1)}{\sum_{r=1}^{N(t-1)} k_{\text{in},r}(t-1)}, \quad (2)$$

where $k_{\text{in},j}(t-1)$ means the in-degree number of edges of the vertex j at time $t-1$. We assume in the model that each new vertex is given an incoming edge pointed from itself, otherwise the in-degree number never grows with time. (iii) each vertex updates its outgoing edges by either adding new edges or deleting existing edges through a multiplicative stochastic process. Let $k_{\text{out},i}(t)$ denotes the out-degree number of edges of the vertex i at time t . Then $k_{\text{out},i}(t)$ evolves as

$$k_{\text{out},i}(t+1) = k_{\text{out},i}(t)[1 + \zeta_i(t+1)], \quad (3)$$

where $\zeta_i(t)$ means the growth rate of the out-degree number $k_{\text{out},i}(t)$ at time t , which fluctuates from time to time about mean g_i ,

$$\zeta_i(t) = g_i + \xi_i(t), \quad (4)$$

where $\xi_i(t)$ is assumed to be a white noise satisfying $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle = \sigma_i^2 \delta_{i,t} \delta_{i,j}$, where σ_i^2 is the variance. The growth rate g_i and the standard deviation σ_i could vary in general for different vertices. HA, however, assumed that $\{\zeta_i\}$ are uniform for different vertices, i.e., $g_i = g$ and $\sigma_i = \sigma$ for all i . When $\zeta_i(t+1) > 0$, the out-degree number at vertex i is increased. Then we add $k_{\text{out},i}(t)\zeta_i(t+1)$ new edges to the vertex i , pointing to other distinct vertices which are not connected, following the PA rule given by Eq. (2). When $\zeta_i(t+1) < 0$, we delete $k_{\text{out},i}(t)|\zeta_i(t+1)|$ outgoing edges from the vertex i randomly.

III. THE OUT-DEGREE DISTRIBUTION

The out-degree distribution $P_{\text{out}}(k_{\text{out}})$ can be obtained by following the argument given by HA. The conditional probability $P_{\text{out}}(k_{\text{out}}, \tau | m)$ that $k_{\text{out},i} = k_{\text{out}}$ at time $t = t_i + \tau$ for a vertex i born at $t = t_i$, with $k_{\text{out},i} = m$ is given by

$$P_{\text{out}}(k_{\text{out}}, \tau | m) = \frac{1}{k_{\text{out}} \sqrt{2\pi\sigma_0^2\tau}} \exp\left\{-\frac{[\ln(k_{\text{out}}/m) - g_0\tau]^2}{2\sigma_0^2\tau}\right\}. \quad (5)$$

The above distribution was obtained by applying the central limit theorem for the variable $\ln(k_{\text{out}}(t)/k_{\text{out}}(t-1))$, so that g_0 and σ_0^2 in Eq. (5) are related to g and σ^2 as $g_0 \approx g - \sigma^2/2$, and $\sigma_0^2 \approx \sigma^2$ respectively [21]. Since the density of vertices with age τ is proportional to $\rho(\tau) \sim \exp(-p\tau)$, the out-degree distribution collected over all ages becomes

$$P_{\text{out}}(k_{\text{out}}) = \int d\tau \rho(\tau) P_{\text{out}}(k_{\text{out}}, \tau | m) \sim k_{\text{out}}^{-\gamma_{\text{out}}}, \quad (6)$$

where

$$\gamma_{\text{out}} = 1 - \frac{g_0}{\sigma_0^2} + \frac{\sqrt{g_0^2 + 2p\sigma_0^2}}{\sigma_0^2}. \quad (7)$$

We note that the out-degree exponent γ_{out} depends on the three parameters, p , g_0 , and σ_0 .

IV. THE IN-DEGREE DISTRIBUTION

The in-degree number of edges at a vertex i is increased as new edges are additionally pointed from other vertices to i , or decreased as already connected edges are deleted from other vertices. For the increased case, there are two types of occasions. The first is the case in which some of edges from newly born vertices are connected to the vertex i . Since the total number of edges generated from new vertices at time t is given by

$$\mathcal{L}_{\text{new}}(t) = mpN(t-1), \quad (8)$$

the in-degree number of edges of the vertex i evolves as

$$\frac{\partial k_{\text{in},i}(t)}{\partial t} = \frac{k_{\text{in},i}(t-1)}{N(t-1)} \mathcal{L}_{\text{new}}(t) - \sum_{r=1} k_{\text{in},r}(t-1) \quad (9)$$

Second is the case in which the vertex i receives edges from existing vertices as they update their outgoing edges. The total number of newly added outgoing edges is given by

$$\mathcal{L}_{\text{add}}(t) = \sum_{j=1}^{N(t-1)} k_{\text{out},j}(t-1) \zeta_j^+(t), \quad (10)$$

where $\zeta_j^+(t)$ denotes the one when $\zeta_j(t) > 0$. Then, the in-degree number of edges of the vertex i evolves as

$$\frac{\partial k_{\text{in},i}(t)}{\partial t} = \frac{k_{\text{in},i}(t-1)}{N(t-1)} \mathcal{L}_{\text{add}}(t) - \sum_{r=1} k_{\text{in},r}(t-1) \quad (11)$$

On the other hand, the decreased case occurs when other vertices remove their connections to the vertex i . This case occurs when $\zeta_j(t) < 0$ for a vertex $j \neq i$, with $\zeta_j(t)$ denoted by $\zeta_j^-(t)$. The total number of edges removed through this updating process is

$$\mathcal{L}_{\text{del}}(t) = \sum_{j=1}^{N(t-1)} k_{\text{out},j}(t-1) |\zeta_j^-(t)|. \quad (12)$$

Although the edges deleted are chosen randomly, the vertex with a higher in-degree number of edges has more incoming edges deleted because incoming edges were formed following the PA rule. Thus the deletion process leads to

$$\frac{\partial k_{\text{in},i}(t)}{\partial t} = \frac{k_{\text{in},i}(t-1)}{N(t-1)} \mathcal{L}_{\text{del}}(t) - \sum_{r=1} k_{\text{in},r}(t-1) \quad (13)$$

Altogether the dynamic equation for the in-degree number of edges of the vertex i is written as

$$\frac{\partial k_{\text{in},i}(t)}{\partial t} = \frac{k_{\text{in},i}(t-1)}{N(t-1)} [\mathcal{L}_{\text{new}}(t) + \mathcal{L}_{\text{add}}(t) - \mathcal{L}_{\text{del}}(t)] - \sum_{r=1} k_{\text{in},r}(t-1) \quad (14)$$

The above equation can be rewritten as

$$\frac{\partial k_{\text{in},i}(t)}{\partial t} = k_{\text{in},i}(t-1) \left(\frac{mpN(0)e^{pt}}{L(t)} + g_0 + \frac{\chi(t)}{L(t)} \right), \quad (15)$$

where $L(t)$ denotes the total number of incoming edges at time t ,

$$L(t) = \sum_i^{N(t)} k_{\text{in},i}(t), \quad (16)$$

which behaves asymptotically as

$$L(t) \approx \begin{cases} A_1 e^{pt} & \text{if } p > g_0, \\ A_2 t e^{pt} & \text{if } p = g_0, \\ A_3 e^{g_0 t} & \text{if } p < g_0, \end{cases} \quad (17)$$

where A_1 , A_2 , and A_3 are given as

$$A_1 = \frac{mpN(0)}{(p-g_0)}, \quad (18)$$

$$A_2 = mpN(0), \quad (19)$$

and

$$A_3 = \frac{mpN(0)}{(g_0-p)}. \quad (20)$$

$\chi(t)$ in Eq. (15) is defined as

$$\chi(t) = \sum_i^{N(t-1)} k_{\text{out},i}(t-1) (\xi_i^+ - |\xi_i^-|), \quad (21)$$

where $\xi_i^+(t)$ [$\xi_i^-(t)$] denotes the noise for $\xi_i(t) > 0$ [$\xi_i(t) < 0$]. Then using the stochastic property $\langle \xi_i \rangle = 0$, we obtain that

$$\langle \chi(t) \rangle = 0, \quad (22)$$

and

$$\langle \chi(t) \chi(t') \rangle \approx \begin{cases} B_1 e^{pt} \delta_{t,t'} & \text{if } p > 2(g_0 + \sigma_0^2/2), \\ B_2 t e^{pt} \delta_{t,t'} & \text{if } p = 2(g_0 + \sigma_0^2/2), \\ B_3 e^{2(g_0 + \sigma_0^2/2)t} \delta_{t,t'} & \text{if } p < 2(g_0 + \sigma_0^2/2), \end{cases} \quad (23)$$

where B_1 , B_2 , and B_3 are given by

$$B_1 = \frac{m^2 \sigma_0^2 p N_0}{2(p - \sigma_0^2 - 2g_0)}, \quad (24)$$

$$B_2 = m^2 \sigma_0^2 p N_0 (1 + p)^2, \quad (25)$$

and

$$B_3 = \frac{m^2 \sigma_0^2 p N_0}{2(\sigma_0^2 + 2g_0 - p)}. \quad (26)$$

Thus $\chi(t)$ plays a role of noise, and its variance depends on time.

The asymptotic behavior of the dynamic equation Eq. (15) depends on relative magnitudes among p , g_0 , and σ_0^2 . We consider every possible case below.

(i) When $p \geq g_0 + \sigma_0^2/2$ (i.e., $p \geq g$), the stochastic term, the last term in Eq. (15), is negligible in long time limit. Moreover, since $N(t)$ and $L(t)$ have the same time dependence, Eq. (15) is simply reduced to

$$\frac{\partial k_{in,i}(t)}{\partial t} = p k_{in,i}(t). \quad (27)$$

Thus the in-degree number of edges of a vertex i born at time $t = t_i$ becomes

$$k_{in,i}(t) = e^{p(t-t_i)}. \quad (28)$$

Then the in-degree distribution becomes

$$P_{in}(k_{in}) = \frac{\partial}{\partial k_{in,i}} [1 - P(k_{in} > k_{in,i})] \Big|_{k_{in,i}=k_{in}} \quad (29)$$

$$= \frac{\partial}{\partial k_{in,i}} \left(-p \frac{m}{k_{in,i}} \right) \Big|_{k_{in,i}=k_{in}} \propto k_{in}^{-\gamma_{in}}, \quad (30)$$

with $\gamma_{in} = 2$.

(ii) When $g_0 \leq p < g_0 + \sigma_0^2/2$ (i.e., $g - \sigma^2/2 \leq p < g$), the dynamic equation is reduced to asymptotically

$$\frac{\partial k_{in,i}(t)}{\partial t} = k_{in,i}(t) \left(p + \frac{\chi(t)}{L(t)} \right). \quad (31)$$

Since $\langle \chi(t) \rangle = 0$, one may regard the above equation as a stochastic log-normal dynamic equation with the variance,

$$\frac{\langle \chi(t) \chi(t') \rangle}{L(t)^2} = \mathcal{D}_1^2 e^{2st} \delta_{t,t'}, \quad (32)$$

with $\mathcal{D}_1^2 = B_3/A_1^2$ and $s = g_0 + \sigma_0^2/2 - p$. Since $s > 0$, the fluctuation term cannot be ignored. Invoking the central limit theorem, the conditional probability $P_{in}(k_{in}, \tau | k_{in,0})$, that $k_{in,i} = k_{in}$ at time $t = t_i + \tau$, given $k_{in,i} = k_{in,0} = 1$ at $t = t_i$ becomes

$$P_{in}(k_{in}, \tau | k_{in,0}) = \frac{1}{k_{in} \sqrt{2\pi \mathcal{D}_1^2 (e^{2s\tau} - 1)/2s}} \times \exp \left\{ - \frac{[\ln(k_{in}/k_{in,0}) - p\tau]^2}{2\mathcal{D}_1^2 (e^{2s\tau} - 1)/2s} \right\}. \quad (33)$$

So the in-degree distribution can be obtained through

$$P_{in}(k_{in}) = \int_0^t \rho(\tau) P_{in}(k_{in}, \tau | k_{in,0}) d\tau. \quad (34)$$

When $2st \gg 1$, it can be shown using the saddle point approximation that the in-degree distribution is of the form,

$$P_{in}(k_{in}) \sim \frac{1}{k_{in} (\ln k_{in})^{(p/s+1)}}, \quad (35)$$

which is valid as long as $\ln k_{in} \ll e^{2st}$. When $p = g_0$, \mathcal{D}_1^2 is replaced by B_3/A_2^2 .

(iii) For $p < g_0$ (i.e., $p < g - \sigma^2/2$), the dynamic equation of $k_{in,i}(t)$ can be written as

$$\frac{\partial k_{in,i}(t)}{\partial t} = k_{in,i}(t) \left(g_0 + \frac{\chi(t)}{L(t)} \right). \quad (36)$$

The variance can be written as

$$\frac{\langle \chi(t) \chi(t') \rangle}{L(t)^2} = \mathcal{D}_2^2 e^{\sigma_0^2 t} \delta_{t,t'}, \quad (37)$$

with $\mathcal{D}_2^2 \equiv B_3/A_3^2$. Following the same step as used in the second case, we obtain that

$$P_{in}(k_{in}) \sim \frac{1}{k_{in} (\ln k_{in})^{(2g_0/\sigma_0^2+1)}}. \quad (38)$$

In short, when the growth rate of vertex p is larger than the effective growth rate of edge, $g_0 + \sigma_0^2/2$, the in-degree distribution is independent of the detail of evolving networks, so that the in-degree exponent is robust for different systems, while the out-degree exponent depends on the detail. This is the case we observe in the real WWW because the number of webpages increases rapidly nowadays, whereas average number of hyperlinks does rather at a slower rate due to limited space on webpage. When the number of webpages is saturated in the future, the growth rate p will become moderated with the number of hyperlinks g_0 much dominant. Then the in-degree distribution exhibits a phase transition to the form Eq. (35) or (38), implying that the hyperlink is much centralized to a few famous webpages. The phase diagram is depicted in Fig. 1.

V. NUMERICAL SIMULATIONS

It was reported [17] that the Web consisted of $N = 203 \times 10^6$ documents from the viewpoint of Altavista, and the

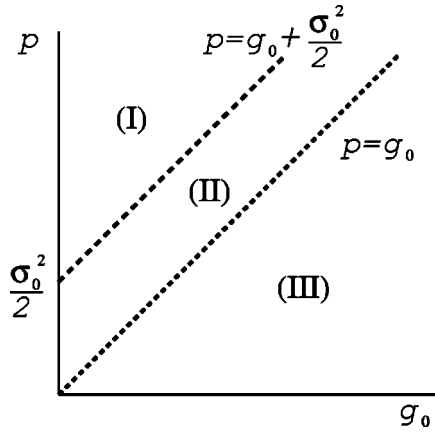


FIG. 1. Schematic phase diagram of various behaviors of the in-degree distribution in the parameter space (p, g_0) .

average in-degree and out-degree number of edges are $\bar{k}_{in} = \bar{k}_{out} = 7.22$ as of May of 1999, and $N \approx 271 \times 10^6$ and $\bar{k}_{in} = \bar{k}_{out} = 7.85$ as of October of 1999. Based on this data, we estimate very roughly the vertex and the mean degree growth

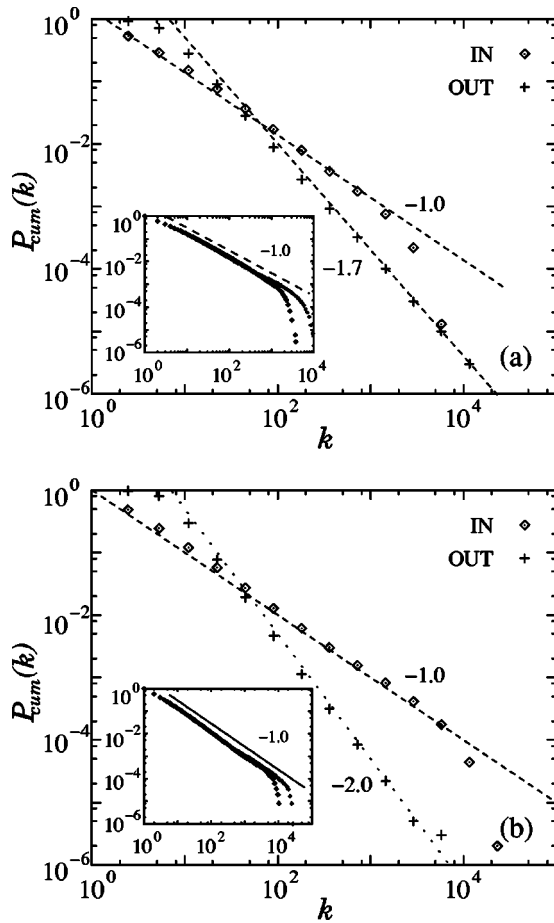


FIG. 2. Plot of the in-degree and out-degree distributions drawn in the cumulated way, $P_{cum}(k) = \int_k^\infty P(k)dk$ for the cases, $p = 0.059$, $g = 0.017$, and $\sigma^2 = 0.051$ for (a), and $\sigma^2 = 0.021$ for (b). Both cases belong to the region (I). Insets: The in-degree distributions for different system sizes, $N = 319\,383$ and $1\,679\,603$ in (a), and $N = 430\,671$ and $1\,375\,434$ in (b).

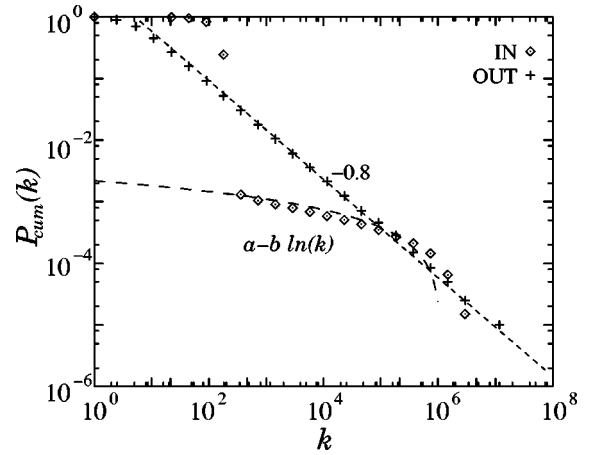


FIG. 3. Plot of the in-degree and out-degree distributions drawn in the cumulated way, $P_{cum}(k) = \int_k^\infty P(k)dk$, for the cases $p = 0.010$, $g = 0.017$, and $\sigma^2 = 0.051$, belonging to the region (II). The numerical data for the out-degree distribution show the power-law behavior with the exponent $\gamma_{out} \approx 1.8$, and those for the in-degree distribution with $\gamma_{in} \approx 1$. $a = 0.0022$ and $b = 0.00016$.

rates to be $p \approx 0.059$ and $g \approx 0.017$ per month, respectively. However, the fluctuation strength σ is not known. Using the estimated values of p and g , we perform numerical simulations for the stochastic model following the HA idea, where the variances σ_0^2 are chosen to be in the regions (I) and (II). The simulation results are compared with the theoretical predictions. First, we choose the variances $\sigma_0^2 = 0.052$ and 0.021 belonging to the region (I). As seen in Fig. 2, the in-degree exponents γ_{in} are robust to be $\gamma_{in} \approx 2$ for both cases, while the out-degree exponents γ_{out} are different from each other as $\gamma_{out} \approx 2.7$ for $\sigma_0^2 \approx 0.052$ and $\gamma_{out} \approx 3.0$ for $\sigma_0^2 \approx 0.021$. The simulation results are close to the theoretical predictions according to Eq. (7), $\gamma_{out} \approx 2.7$ and $\gamma_{out} \approx 3.1$, respectively. Note that the deviation of the numerical simulation data from the analytic solution for large k_{in} are due to finite system size effect. To see this, in the insets of Fig. 2, we show the in-degree distributions for different system sizes for each case. Indeed, the power-law region is more extended for larger system sizes for both cases of Figs. 2(a) and 2(b). Second, we choose $p = 0.010$, $g = 0.017$, and $\sigma_0^2 = 0.041$ belonging to the region (II). The power-law behavior for the in-degree distribution appears for large k_{in} with the exponent $\gamma_{in} \approx 1$ as shown in Fig. 3, in agreement with the theoretical prediction without the logarithmic correction. For the out-degree distribution, the power-law behavior is also obtained with the exponent $\gamma_{out} \approx 1.8$, which is in agreement with the theoretical value 1.8 according to Eq. (7).

VI. CONCLUSIONS

We have introduced a stochastic model for directed SF networks, which evolves with time. In our model, the evolution of outgoing edges follows the stochastic multiplicative

process, while that of incoming edges does the preferential attachment. With this model, we could illustrate why the in-degree exponent for the WWW is robust, independent of different systems, while the out-degree exponent depends on different systems. We presented analytic results for both the in-degree and the out-degree distributions and confirmed our theoretical predictions by performing numerical simulations

with the parameter values estimated from the WWW in real world.

ACKNOWLEDGMENTS

We would like to thank A.-L. Barabási for helpful discussions. This work was supported by the Korean Research Foundation (Grant No. 01-041-D00061).

-
- [1] S.H. Strogatz, *Nature (London)* **410**, 268 (2001).
 - [2] N. Goldenfeld and L.P. Kadanoff, *Science* **284**, 87 (1999).
 - [3] P. Erdős and A. Rényi, *Publ. Math.* **5**, 17 (1960).
 - [4] A.-L. Barabási, R. Albert, and H. Jeong, *Physica A* **272**, 173 (1999); R. Albert and A.-L. Barabási, e-print cond-mat/0106144.
 - [5] A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999).
 - [6] R. Albert and A.-L. Barabasi, *Phys. Rev. Lett.* **85**, 5234 (2000).
 - [7] G. Bianconi and A.-L. Barabási, *Europhys. Lett.* **54**, 436 (2001).
 - [8] B.A. Huberman and L.A. Adamic, *Nature (London)* **401**, 131 (1999); B.A. Huberman and L.A. Adamic, e-print cond-mat/9901071.
 - [9] M.E.J. Newman, S.H. Strogatz, and D.J. Watts, *Phys. Rev. E* **64**, 026118 (2001).
 - [10] M.E.J. Newman, *Proc. Natl. Acad. Sci. U.S.A.* **98**, 404 (2001); L.A. Amaral, A. Scala, M. Barthélémy, and H.E. Stanley, *ibid.* **97**, 11149 (2000).
 - [11] E.W. Zegura, K.L. Calvert, and M.J. Donahoo, *IEEE/ ACM Trans. Network* **5**, 770 (1997); M. Faloutsos, P. Faloutsos, and C. Faloutsos, *Comput. Commun. Rev.* **29**, 251 (1999).
 - [12] R. Albert, H. Jeong, and A.-L. Barabasi, *Nature (London)* **401**, 130 (1999).
 - [13] R. P. Kumar, S. Raghavan, D. Rajalopagan, A. S. Tomkins, and E. Upfal, in *Proceedings of the IEEE FOCS 2000* (IEEE Computer Society, Los Alamitos, CA, 2000), pp. 57–65.
 - [14] S. Redner, *Eur. Phys. J. B* **4**, 131 (1998).
 - [15] H. Jeong, B. Tombor, R. Albert, Z.N. Oltvani, and A.-L. Barabási, *Nature (London)* **407**, 651 (2000).
 - [16] J. M. Kleinberg, *Nature (London)* **406**, 845 (2000).
 - [17] A. Broder, R. Kumar, F. Maghoul, P. Raghavan, R. Sridhar, R. Stata, A. Tomkins, and J. Wiener, *Computer Networks* **33**, 209 (2000).
 - [18] S.N. Dorogovtsev and J.F.F. Mendes, e-print cond-mat/0106144.
 - [19] P.L. Krapivsky, G.J. Rodgers, and S. Redner, *Phys. Rev. Lett.* **86**, 5401 (2001).
 - [20] B. Tadić, *Physica A* **293**, 273 (2001).
 - [21] K.-I. Goh, B. Kahng, and D. Kim, *Phys. Rev. Lett.* **88**, 108701 (2002).